Solution 11

16.7 no 13. Solution The surface S is a parabolic cone whose base is the disk of radius 2 at the origin and tip at at $(0,0,4)$. By Stokes' theorem, the flux of the curl of $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, that is,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

is equal to

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

As S is oriented by the outer normal, the boundary of S, C, is in anticlockwise direction.

The boundary curve is the circle $x^2 + y^2 = 4$. A standard parametrization is $\mathbf{c}(\theta) = 2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j} + 0\mathbf{k},\ \theta \in [0, 2\pi]$. We have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta)\, d\theta \\
&= \int_0^{2\pi} (8\mathbf{i} + 6\cos\theta\mathbf{j} + 10\sin\theta\mathbf{k})(-2\sin\theta\mathbf{i} + 2\cos\theta\mathbf{j})\, d\theta \\
&= 12\int_0^{2\pi} \cos^2\theta\, d\theta \\
&= 12\pi.
\end{aligned}$$

16.7 no 16. Solution It suffices to point out S is the cone described by $z = 5 - \sqrt{x^2 + y^2}$ so that its base is a disk of radius 5 in the xy-plane. As the normal on S is pointing outward, a good parametrization of its boundary C is

$$\mathbf{c}(\theta) = 5\cos\theta\mathbf{i} + 5\sin\theta\mathbf{j} + 0\mathbf{k}\ , \theta \in [0, 2\pi].$$

The rest is routine.

16.7 no 18. Solution It suffices to point out S is the upper hemisphere of radius 2 whose boundary C is a circle sitting in the xy-plane. As the normal on S is pointing outward, a good parametrization of C is

$$\mathbf{c}(\theta) = 2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j} + 0\mathbf{k}\ , \theta \in [0, 2\pi].$$

The rest is routine.

Supplementary Problems

1. Let S be the triangle with vertices at $(1,0,0), (0,2,0), (0,0,7)$ with normal pointing upward. Find the circulation of the vector field $\mathbf{F} = x\mathbf{i} + 3z\mathbf{j}$ around the boundary of S with the orientation determined by the chosen normal of S.

   Solution. The equation for the triangle is given by $ax + by + cz = d$. To determine the coefficients, we note $(0,2,0) - (1,0,0) = (-1,2,0)$ and $((0,0,7) - (1,0,0) = (-1,0,7)$ and the vector $(-1,2,0) \times (-1,0,7) = (14,7,2)$ is in the normal direction of the surface. So the equation is given by $14x + 7y + 2z = d$. Since $(1,0,0)$ is on the surface, $14 \times 0 + 0 + 0 = d$, $d = 14$. The equation of the plane containing the triangle is $14x + 7y + 2z = 14$. The normal pointing upward, that is, the z-component is positive, is $(14,7,2)$.

We have $\nabla \mathbf{F} = -3\mathbf{i}$. The surface is the graph of $z = (14 - 14x - 7y)/2$ over the triangle T with vertices at $(0,0), (1,0), (0,2)$. Then $\mathbf{r}_x = (1,0,-7)$ and $\mathbf{r}_y = (0,1,-7/2)$ so $\mathbf{r}_x \times \mathbf{r}_y = 7\mathbf{i} + 7/2\mathbf{j} + \mathbf{k}$. We have

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_T (-3\mathbf{i}) \cdot (7\mathbf{i} + 7/2\mathbf{j} + \mathbf{k})\, dA \\ &= -21 \iint_T dA \\ &= -21.\end{aligned}$$

We conclude

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = -21. \tag{1}$$

Hence by Stokes' theorem, the circulation of $\mathbf{F}$ is equal to -21.

Note. A direct verification. Tthe boundary of S consists of three line segments C_1, C_2, C_3. First, C_1 from $(1,0,0)$ to $(0,2,0)$ which is parametrized by $(x(t), y(t), z(t)) = (1,0,0) + t(-1,2,0) = (1-t, 2t, 0), t \in [0,1]$. We have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (x(t)\mathbf{i} + 3z(t)\mathbf{j}) \cdot (-\mathbf{i} + 2\mathbf{j})\, dt = -\int_0^1 (1-t)\, dt = -1/2\ .$$

Next, C_2 from $(0,2,0)$ to $(0,0,7)$ is parametrized by $(0, 2-27, 7t), t \in [0,1]$. As before we get

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -21\ .$$

Finally, C_3 from $(0,0,7)$ to $(1,0,0)$ is parametriized by $(t, 0, 7-7t), t \in [0,1]$. We have

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 1/2\ .$$

It follows that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) \mathbf{F} \cdot d\mathbf{r} = -1/2 - 21 + 1/2 = -21\ ,$$

which is equal to (1).

2. Show that for a closed oriented surface S, that is, a surface without boundary,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0\ .$$

Hint: See how to apply Stokes' theorem.

Solution. Choose a point on S and draw a sphere with center at this point. When the radius of the sphere is sufficiently small, the intersection of the sphere with S is a simple closed curve C. Call the part of S outside C S_1 and the inside S_2. Then S is the union of S_1 and S_2 with common boundary C. Denote the oriented boundary with the correct orientation with S_1 by C and the oriented boundary with the correct direction with S_2 be C'. C and C' are the same curve but with opposite orientation. We have

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \left(\iint_{S_1} + \iint_{S_2}\right) \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C'} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} - \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= 0\ .\end{aligned}$$

3. (Optional) Let S be the surface given by $(x, y) \mapsto (x, y, f(x, y)), (x, y) \in D$. That is, it is the graph of f over the region D. Show that in this case Stokes' theorem

$$\iint_S \nabla \times \mathbf{F} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} \ ,$$

($\mathbf{F}$ is a smooth vector field on S) can be deduced from Green's theorem for some vector field on D. Hint: Let the boundary of D be $\mathbf{r}(t) = (x(t), y(t))$. Then the boundary of S is $\mathbf{c}(t) = (x(t), y(t), f(x(t), y(t)))$. Convert the integration in S and C to the integration on D and the boundary of D respectively.

Solution. Use $(x, y) \mapsto (x, y, f(x, y))$ to parametrize S (what else?). The upward normal is given by

$$\mathbf{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}} \ ,$$

so

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_D [(P_y - N_z)(-f_x) - (P_x - M_z)(-f_y) + (N_x - M_y)] \, dA(x, y) \ , \quad (2)$$

where the curl of $\mathbf{F}$ is evaluated at $(x, y, f(x, y))$. On the other hand, let γ be the boundary of D parametrized by $(x(t), y(t)), t \in [a, b]$. Then C is parametrized by $(x(t), y(t), f(x(t), y(t))$, so its velocity is $(x'(t), y'(t), f_x x'(t) + f_y y'(t))$. We have

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b [Mx'(t) + Ny'(t) + P(f_x x'(t) + f_y y'(t))] \, dt \\ &= \int_a^b [(M + Pf_x)x'(t) + (N + Pf_y)y'(t)] \, dt \ , \end{aligned}$$

where $\mathbf{F}$ is evaluated at $(x(t), y(t), f(x(t), y(t)))$. Letting $R = M + Pf_x$ and $Q = N + Pf_y$ (evaluating at $(x, y, f(x, y))$ and applying Green's theorem to the vector field $R\mathbf{i} + Q\mathbf{j}$ on D, we have

$$\begin{aligned} & \int_a^b [(M + Pf_x)x'(t) + (N + Pf_y)y'(t)] \, dt \\ = & \oint_\gamma (Rx'(t) + Qy'(t)) \, dt \\ = & \oint_\gamma (R\mathbf{i} + Q\mathbf{j}) \cdot d\mathbf{r} \\ = & \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y}) \, dA \\ = & \iint_D [(N_x + N_z f_x + (P_x + P_z f_x) f_y + P f_{yx}) - (M_y + M_z f_y + (P_y + P_z f_y) f_x + P f_{xy})] \, dA \\ = & \iint_D (N_x + N_z f_x + P_x f_y - M_y - M_z f_y - P_y f_x) \, dA \ , \end{aligned}$$

which is equal to (2). Stokes' theorem holds in this case.

Note. The general result can be obtained by cutting the surface into finitely many small pieces so that they are graphs over the xy- or other planes. Applying the above result to obtain the Stokes' formula on each piece. By adding these formulas up we get the Stokes' formula for the entire surface, noting that most of the boundaries of the small pieces cancel out.